Simple absorbing-state transition

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We study a simple reaction-diffusion process that exhibits a phase transition to an absorbing phase in its steady state. We characterize the universal properties of the transition by computing the associated critical exponents. We suggest that the exclusion constraint between particles may change the universality class of the transition even though the density is asymptotically low at the transition. This is suprising as no segregation or jamming phenomena are in play since we are dealing with a single species diffusing without drift.

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I. MOTIVATIONS

Absorbing-state transitions in nonequilibrium steady states occur when a system evolving in time gets trapped in a state that it cannot escape from [1]. In many examples absorbing states are completely inactive configurations in which all microscopic degrees of freedom remain constant throughout time. Such absorbing states cannot be encountered in equilibrium systems. On the experimental side, transitions to an absorbing state were first found in chemical reactions (the Schlögl autocatalytic reaction). From the theoretical standpoint, they are natural nonequilibrium generalizations of the phase transitions between Gibbs states found in equilibrium critical phenomena. The paradigmatic example of such a transition is embodied by the so-called directed percolation (DP) universality class. The latter has been invoked to describe surface roughening, fluid flows in porous media, cellular automata [2], epidemic spreading in population dynamics models, avalanches in some self-organized sand-pile models [3], and a host of reaction-diffusion processes. It remains the focus of intense experimental interest [4] because, up to this day, there has been no experimental confirmation of the critical exponents found by numerical simulations of lattice models. Of less general relevance but of equal theoretical importance, we mention the branching and annihilating random walk with an even number of offsprings (BARWE) whose universal properties were recently elucidated by Cardy and Täuber [5]. Recently, Rossi *et al.* [6] unravelled another universality class in performing an extensive exploration of stochastic fixed-energy sand-pile models (FES) and of related stochastic processes. In the following, we will always adopt the reaction-diffusion process vocabulary; the microscopic degrees of freedom are local particle numbers. In that language, DP is characterized by the two reactions $A\to A+A$, $A+A\to\varnothing$ with diffusion, BARWE by $A\to A+A+A$, $A+A\to\varnothing$ with diffusion, and an example of a system in the FES class is one in which pairs of particles can perform independent nearest-neighbor jumps (a single particle on a site remains at rest; the order parameter is the number of pairs of particles sitting on the same site). We emphasize that almost no exact result is known on those systems from an analytic viewpoint [7]. This makes simple models particularly welcome and the one we shall introduce in this paper already shares most of the difficulties of directed percolation.

There are a handful of microscopic ingredients that are well known to change the universality class of a transition, often inspired by our knowledge of equilibrium critical phenomena. Noise with long-range correlations or long-range jumps (of a Lévy flight type), boundary effects [8], or quenched disorder are three such ingredients. Another ingredient that is specific to nonequilibrium steady states is the number of absorbing states the system can fall into. It is unique in the DP case. There are two in the BARWE case: the parity of the total number of particles being conserved throughout time, depending on the initial number of particles, the absorbing state will have either one or zero particle. In the FES-related cases, where systems possess an additional conservation law, there is an infinite number of absorbing states that the system can freeze into. We refer the reader to the recent review of Hinrichsen [9].

In the present paper, because the DP class is the most prominent of all by far, we focus on an extremely simplified version of a DP-like process on which we wish to test the importance of using mutually excluding particles (or *fermionic*) versus nonexcluding (or *bosonic*) ones on the universal properties of the system. Analytic progress will be made possible due to the simplicity of our system. There is an increasing body of evidence that the hardcore constraint may change critical properties [10]. This has stimulated us to look at a model simple enough for analytic statements to be made.

We have organized our work as follows. in Sec. II, we give a precise description of the microscopic rules of our model and provide further motivations for introducing it. Section III is dedicated to a renormalization group analysis of the critical properties of the process. This includes a derivation of the renormalized equation of state from which we can extract the order parameter without resorting to a scaling hypothesis. We devote Sec. IV to giving arguments in favor of a difference between bosons and fermions. Finally, we present in an Appendix a formulation of our reaction-diffusion process in terms of a zero-dimensional process with long-term memory (which allows an alternative derivation of the results presented in Sec. III).

II. MODEL AND PHASE DIAGRAM

A. Fermions versus bosons

We consider a single species of particles whose members are generically denoted by *A*. We immediately make the difference between mutually excluding particles that evolve respecting the hardcore constraint. Such particles will abusively be christened *fermionic* particles. We will also need so-called *bosonic* particles, for which there is no hardcore constraint and that do not exclude each other. In the former case, local particle number are restricted to 0 and 1, while they may take any integer value from 0 to ∞ in the latter. The microscopic rules depend slightly on whether we choose fermionic or bosonic particles.

Particles are initially randomly and independently distributed on the sites of a one-dimensional lattice (of unit spacing) with an initial average density ρ_0. Here are the microscopic rules of our reaction-diffusion process.

(1) *Diffusion*. Particles perform a simple isotropic diffusive motion with diffusion constant D. If we choose the fermionic formulation, jumps on nearest-neighbor sites can only occur if the target site is empty, else the motion is rejected (but time keeps elapsing).

(2) *Branching with one offspring*. A particle that sits at the origin of the lattice produces an offspring between t and $t+dt$ with probability λdt. In the bosonic formulation, the offspring is placed at the origin, while in the fermionic formulation, the offspring is placed randomly on one of the two nearest-neighbor sites, provided the target site is empty (else nothing happens).

(3) *Annihilation*. In the bosonic formulation, two particles sitting at the origin annihilate with probability $k dt$ between t and $t+dt$. Within the fermionic description, a pair of nearest neighbor particles one of which sits at the origin annihilate with probability $k dt$ between t and $t+dt$.

No reaction occurs away from the origin of the lattice. The above rules, supplemented with the initial condition, define a Markov process for the set of local occupation numbers $\mathsf{n}\equiv\{n_i\}$, with $n_i=0,1$ in the fermionic formulation, and $n_i=0,\ldots,\infty$ in the bosonic formulation. This constitutes the reaction-diffusion process that we are now going to study.

B. Phase diagram within a mean-field approach

Our mean-field approach does not distinguish between fermions and bosons. We denote by $\psi(x,t)$ the local density of particles at site x. We write a reaction-diffusion equation for $\psi(x,t)$,

$$\partial_t\psi(x,t)=D\partial_x^2\psi+\lambda\,\delta(x)\,\psi-k\,\delta(x)\,\psi^2, \tag{1}$$

in which one easily recognizes the usual diffusion and reaction terms. Since the reaction rates take nonzero values only on the subspace $x=0$, we have introduced reaction terms localized at $x=0$. The above equation is equivalent to

$$\partial_t\psi-D\partial_x^2\psi=0,\quad 2D\partial_x\psi(0^+,t)+\lambda\,\psi(0,t)-k\psi^2(0,t)=0, \tag{2}$$

with ψ continuous and even in the coordinate x. Hence we are dealing with a diffusion equation with nonlinear boundary conditions. It is straightforward to see that, if there exists a steady-state profile $\Psi(x)=\psi(x,t\to\infty)$, one has

$$\partial_x^2\Psi(x)=0, \tag{3}$$

hence, since Ψ must be even it takes the form $\Psi(x)=A|x|+B$ (A,B constants). The existence of a thermodynamic limit imposes $A=0$, hence a steady state is necessarily homogeneous in space. We now return to the boundary condition at $x=0$ from which we deduce

$$\Psi(x)=\frac{\lambda}{k}. \tag{4}$$

We identify the order parameter exponent defined by $\Psi\sim(\lambda-\lambda_c)^\beta$ as $\beta=1$ and $\lambda_c=0$ within the mean-field approximation.

In the particular case $\lambda=0$ the density at the origin decays as $1/\sqrt{t}$. For $\lambda>0$, the system relaxes exponentially to its steady state with a relaxation time $T_{\mathrm{relax}}\sim\lambda^{-\nu z}$, $z=2$, $\nu=1$.

The phase diagram is that of directed percolation: for $\lambda>0$ there is an active phase that contaminates the whole space, while for $\lambda=0$ the system falls into an absorbing state. Of course, the universality class of the phase transition is different from the DP one.

As a final remark we note that there exist two ways to continue our problem to higher space dimensions. The first one—for which we shall present explicit calculations—consists in considering free diffusion in the whole d-dimensional space but reactions confined to the origin. An alternative continuation consists in allowing reactions to take place on a $(d-1)$-dimensional hyperplane. We shall call the former formulation the isotropic model, while the latter is the anisotropic one.

C. Motivations

In this paragraph, we would like to give the motivations that have led to the definition of our toy-model. As usual in statistical mechanics, the greatest task is to assess the effect of fluctuations and correlations between degrees of freedom. In many systems their importance is seen to increase as space dimension is decreased. On the other hand, one has the intuition that, if there exists an upper critical dimension in our system, it will be lower than that of directed percolation (branching and annihilation takes place anywhere in space, along with diffusion), since we have reduced the sources of reaction noise to a minimum level.

At any finite time, the system is strongly inhomogenous, with the density an *a priori* complicated scaling function of all dimensionless variables built with the parameters (ρ_0, $\lambda-\lambda_c$, x, and t). This is obviously not the case in a standard reaction-diffusion process where reactions may occur anywhere in space: there the system is homogeneous. However, comparing with directed percolation, we keep: a single absorbing state, no parity conservation, no additional conserved quantity. Only translation invariance is broken.

One of our aims in defining the above reaction-diffusion process is to investigate the influence on the universal properties of hardcore versus nonexcluding particles. Hence, in either case, we have to assess the role of fluctuations and correlations, and first determine whether the latter are consistent with our mean-field picture. For instance, it is well-known that if reactions could take place anywhere in space, one would come up with a nontrivial exponent $\beta<1$ (in space dimensions below 4) and a nonzero critical branching rate λ_c (the so-called threshold effect in epidemic modeling). Several questions are in order.

(1) Do fluctuations destroy the mean-field picture? How do fluctuations affect the scaling behavior?

(2) What are the critical exponents characterizing the critical behavior?

(3) What is the difference, if any, between using hardcore or nonexcluding particles on the universal properties?

III. BEYOND MEAN FIELD

A. Ginzburg-Landau criterion

In this paragraph we propose a simple argument to determine the upper critical dimension. Define $\delta\psi(\mathbf{r})\equiv\psi(\mathbf{r})-\langle\psi\rangle$. Let $V_\xi=\xi^d$ denote a *coherence volume* and set $\delta_\xi\psi\equiv(1/V_\xi)\int d^d r\,\delta\psi(\mathbf{r})$, which is the local average of the fluctuations in the region over which there are correlations. Correlations between fluctuations are negligible as long as

$$\langle(\delta_\xi\psi)^2\rangle\ll\langle\psi\rangle^2. \tag{5}$$

We first estimate the left-hand side (lhs). One writes

$$(\delta_\xi\psi)^2=\int\frac{d^d r d^d r'}{V_\xi^2}C(\mathbf{r},\mathbf{r}'), \tag{6}$$

where $C(\mathbf{r},\mathbf{r}')$ is the density autocorrelation function. In a finite volume L^d the total particle number fluctuations $(\Delta N_L)^2$ are of order L^d,

$$(\Delta N_L)^2=\int d^d r d^d r' C(\mathbf{r},\mathbf{r}',\xi)\propto L^d, \tag{7}$$

so that in the limit $L\to\infty C(\mathbf{r},\mathbf{r}')$ must have the scaling form

$$C(\mathbf{r},\mathbf{r}',\xi)=\xi^{-d}\mathcal{F}\left(\frac{\mathbf{r}}{\xi},\frac{\mathbf{r}'}{\xi}\right). \tag{8}$$

Hence one has

$$\langle(\delta_\xi\psi)^2\rangle\propto\xi^{-d}. \tag{9}$$

Besides, one knows from a mean-field analysis that $\langle\psi\rangle\propto\xi^{-1/\nu_{\rm mf}}$, so that the condition (5) for mean field to hold becomes

$$\xi^{-d}\ll\xi^{-2/\nu_{\rm mf}}, \tag{10}$$

which is true for $d\nu_{\rm mf}-2>0$. In the isotropic case where reactions take place on a single point in space, $\nu_{\rm mf}^{-1}=2-d$, so that Eq. (10) requires that $d>\frac{4}{3}$, while in the anisotropic case (reactions confined to a hyperplane), $\nu_{\rm mf}^{-1}=1$ hence $d>2$. This establishes the upper critical dimensions $\frac{4}{3}$ and 2 for the isotropic and anisotropic formulations, respectively, above which mean-field analysis applies. Below d_c space fluctuations are expected to play an important role.

B. Bosonic field theory

In this section we shall present the explicit renormalization group calculation for the isotropic d-dimensional continuation of the model defined in Sec. II. Employing the coherent state formalism (see Ref. [11] for a review) one shows that the dynamical properties of the reaction-diffusion process can be deduced from the action

$$S[\bar{\psi},\psi]=\int d^d r dt[\bar{\psi}(\partial_t+\sigma\delta^{(d)}(\mathbf{r})-\nabla_{\mathbf{r}}^2)\psi + g\delta^{(d)}(\mathbf{r})\bar{\psi}\psi(\psi-\bar{\psi})-\rho_0\delta(t)\bar{\psi}], \tag{11}$$

in which the field ψ can be thought of as a density field. The parameters σ, g are coarse-grained versions of the original reaction rates, with $\sigma\propto\lambda_c-\lambda$. The constant ρ_0 is the initial density of particles. Note that the time reversal symmetry

$$\psi(\mathbf{x},t)\to-\bar{\psi}(\mathbf{r},-t),\quad \bar{\psi}(\mathbf{x},t)\to-\psi(\mathbf{r},-t) \tag{12}$$

allows to focus on a single coupling constant g as in a usual DP process. Denoting by a the lattice spacing, the continuum limit was built by scaling the coarse-grained reaction rates and fields according to

$$[\psi]\sim[\bar{\psi}]\sim a^{-d/2},\quad [\sigma]\sim a^{-1/(2-d)},\quad [g^2]\sim a^{3d-4}, \tag{13}$$

which again indicates that $d_c=\frac{4}{3}$ is the upper critical dimension. In the action we have already omitted a term $g'\int dt d^d r\,\delta^{(d)}(\mathbf{r})(\bar{\psi}\psi)^2$ because g' scales as $a^{2(d-1)}$, which makes it an irrelevant coupling in the vicinity of $d=4/3$.

We set $\varepsilon\equiv4-3d$ and we perform an ε expansion of the critical exponents. This is done by introducing a renormalized mass and a renormalized coupling:

$$\sigma=Z_\sigma\sigma_R,\quad g^2G_\varepsilon=Z_u u_R a^{-\varepsilon}, \tag{14}$$

where

$$G_\varepsilon=\frac{4\Gamma\left(1-\frac{d}{2}\right)^2\Gamma\left(1+\frac{\varepsilon}{2}\right)}{\Gamma(2-d)(4\pi)^{3d/2}}$$

is a dimension-dependent factor that we find convenient to incorporate into the renormalized coupling. The Z factors are determined by requiring that the ε poles be absorbed into the renormalized couplings (we follow the dimensional regularization and minimal subtraction scheme).

C. Renormalization

We first express the free propagator $G(x,x',t)$ of the one-dimensional reaction-diffusion process, which is the solution of

$$\partial_t G(x,x';t)+\sigma\delta(x)G(x,x';t)-\partial_x^2 G(x,x';t) =\delta(x-x')\delta(t) \tag{15}$$

We denote by ω the Laplace variable conjugate to t (and assume $t>0$). In dimension 1 we find that

$$G(x,x';\omega)=G_0(x-x';\omega)-\frac{\sigma}{2\sqrt{\omega}(2\sqrt{\omega}+\sigma)}e^{-\sqrt{\omega}(|x|+|x'|)}, \tag{16}$$

with the notation $G_0(y;\omega)=(1/2\sqrt{\omega})/e^{-\sqrt{\omega}|y|}$ [the Laplace transform of $G_0(y;t)=(1/\sqrt{4\pi t})e^{-y^2/4t}$]. It is not hard to find the Fourier transform of the propagator

$$G(k,k';\omega)=(2\pi)\delta(k+k')\frac{1}{k^2+\omega}-\frac{2\sqrt{\omega}\sigma}{2\sqrt{\omega}+\sigma}\frac{1}{k^2+\omega}\frac{1}{k'^2+\omega}. \tag{17}$$

To first order in $\sigma\to 0$, one finds

$$G(k,k';t)=(2\pi)\delta(k+k')e^{-k^2t}-\sigma\frac{e^{-k^2t}-e^{-k'^2t}}{k'^2-k^2}+O(\sigma^2). \tag{18}$$

The d-dimensional propagator is easily inferred from the above. Its Fourier-Laplace transform reads

$$G(\mathbf{k},\mathbf{k}';\omega)=(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}')\frac{1}{\mathbf{k}^2+\omega}-\frac{\frac{(4\pi)^{d/2}}{\Gamma\left(1-\frac{d}{2}\right)}\omega^{1-d/2}\sigma}{\frac{(4\pi)^{d/2}}{\Gamma\left(1-\frac{d}{2}\right)}\omega^{1-d/2}+\sigma}\frac{1}{\mathbf{k}^2+\omega}\frac{1}{\mathbf{k}'^2+\omega}, \tag{19}$$

so that, to leading order in $\sigma\to 0$, one has

$$G(\mathbf{k},\mathbf{k}';t)=(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}')e^{-\mathbf{k}^2t}-\sigma\frac{e^{-\mathbf{k}^2t}-e^{-\mathbf{k}'^2t}}{\mathbf{k}'^2-\mathbf{k}^2}+O(\sigma^2). \tag{20}$$

This is the free propagator of our theory.

Denoting by $\Gamma^{(m,n)}$ the $m+n$ point vertex function with m,n external $\bar{\psi},\psi$ legs, respectively, we find, to one loop:

$$\Gamma^{(1,1)}(\mathbf{k},\mathbf{k}',\omega)=(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}')(\omega+\mathbf{k}^2)+\sigma\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}\times\int_0^\infty dt\,G(\mathbf{k}_1,\mathbf{k}_2;t)G(\mathbf{k}_3,\mathbf{k}_4;t), \tag{21}$$

which becomes, in the limit $\sigma\to 0$,

$$\Gamma^{(1,1)}(\mathbf{k},\mathbf{k}',\omega)=(2\pi)^d\delta^{(d)}(\mathbf{k}+\mathbf{k}')(\omega+\mathbf{k}^2)+\sigma\left[1-\frac{g^2G_\varepsilon}{\varepsilon}\right]. \tag{22}$$

Similarly, one finds

$$\Gamma^{(1,2)}(\{\mathbf{k}_i=\mathbf{0}\})=2g\left(1-\frac{2g^2G_\varepsilon}{\varepsilon}\right). \tag{23}$$

Thus we find

$$Z_\sigma=1+\frac{u_R}{\varepsilon},\quad Z_u=1+\frac{4u_R}{\varepsilon} \tag{24}$$

and the related Wilson functions,

$$\gamma_\sigma=a\frac{d\ln Z_\sigma}{da}\simeq u_R,\quad \gamma_u=a\frac{d\ln Z_u}{da}\simeq 4u_R. \tag{25}$$

We deduce the coupling β function

$$\beta_u\equiv a\frac{du_R}{da}=u_R(\varepsilon-\gamma_u)\simeq u_R(\varepsilon-4u_R). \tag{26}$$

It has a single stable fixed point $u_R^*=\varepsilon/4$ from which we deduce the correlation length critical exponent

$$\nu^{-1}=2-d-\gamma_\sigma\simeq\frac{2}{3}+\frac{\varepsilon}{12}. \tag{27}$$

It is important to note that the field and the diffusion constant remain unrenormalized, which imposes

$$\frac{\beta}{\nu}=\frac{d}{2},\quad z=2, \tag{28}$$

to all orders in ε (that is, providing there exists a perturbative fixed point). This is an interesting *exact* prediction of the field-theoretic approach that has no equivalent property in usual DP. The order parameter exponent has the expression

$$\beta=1-\frac{3\varepsilon}{8}, \tag{29}$$

which is valid to first order in ε.

Interestingly, in the anisotropic d-dimensional continuation of our model, the critical exponents read $\nu^{-1}=1-\varepsilon'/4$ and $\beta=(2+\varepsilon')/3$, with $\varepsilon'=2-d$. The independent $\varepsilon'=2-d$ and $\varepsilon=4-3d$ expansions coincide to leading order. The property $\beta/\nu=d/2$ is maintained.

D. One loop calculation of the equation of state

We now derive the equation of state for the stationary uniform value of $\psi(\mathbf{r},t)$ in the isotropic case. This will allow us to find the explicit expression of the order parameter in the active phase close to the critical point without resorting to any scaling hypothesis.

We perform the replacement $\overline{\psi}=\overline{\phi}$ and $\psi=\phi+\Psi$ in the action Eq. (11). The quantity Ψ is chosen to be the uniform stationary solution of the equation of state. In terms of the new fields the action reads

$$S[\overline{\phi},\phi,\Psi]=\int dt\int d^dr\{\overline{\phi}(\partial_t+\delta^{(d)}(\mathbf{r})(\sigma+2g\Psi)-\Delta_{\mathbf{r}})\phi -g\overline{\phi}^2\Psi+\delta^{(d)}(\mathbf{r})[k(\overline{\phi}\phi)^2+g^2\overline{\phi}\phi(\phi-\overline{\phi})] +\overline{\phi}[\partial_t+\delta^{(d)}(\mathbf{r})\sigma-\Delta_{\mathbf{r}}]\Psi-\delta^{(d)}(\mathbf{r})g\overline{\phi}\Psi +\delta^{(d)}(\mathbf{r})[2k\overline{\phi}^2\phi\Psi+k(\overline{\phi}\Psi)^2+g\overline{\phi}\Psi^2]\}. \tag{30}$$

The equation of state is obtained by writing that

$$\frac{\delta\Gamma}{\delta\overline{\phi}}[\phi=0,\overline{\phi}=0]=0. \tag{31}$$

To one loop order the lhs of Eq. (31) reads

$$\sigma\Psi+g\Psi^2-g\int\frac{d^dq_1}{(2\pi)^d}\int\frac{d^dq_2}{(2\pi)^d}\langle\phi(\mathbf{q}_1,t)\phi(\mathbf{q}_2,t)\rangle=0, \tag{32}$$

where we have used the fact that Ψ is constant in space and time. The initial condition was projected back to $t=-\infty$. Of course Eq. (32) has the solution $\Psi=0$. We now need the density of particles autocorrelation function

$$C(\mathbf{q}_1,t;\mathbf{q}_2,t')\equiv\langle\phi(\mathbf{q}_1,t)\phi(\mathbf{q}_2,t')\rangle. \tag{33}$$

Standard techniques of Gaussian integrations yield

$$C(\mathbf{q}_1,t;\mathbf{q}_2,t')=2g\Psi\int_0^{t_<}ds\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\times G_\Psi(\mathbf{q}_1,\mathbf{k}_1;t-s)G_\Psi(\mathbf{q}_2,\mathbf{k}_2;t'-s), \tag{34}$$

where the index G_Ψ means exactly the function G_0 of Eq. (19) in which σ was replaced with $\sigma_\Psi\equiv\sigma+2\Psi$, that is,

$$G_\Psi(\mathbf{q},\mathbf{q}',\omega)=(2\pi)^d\delta^{(d)}(\mathbf{q}+\mathbf{q}')\frac{1}{\mathbf{q}^2+\omega}-\frac{\sigma_\Psi C_d\omega^{1-d/2}}{\sigma_\Psi+C_d\omega^{1-d/2}}\frac{1}{\mathbf{q}^2+\omega}\frac{1}{\mathbf{q}'^2+\omega}. \tag{35}$$

Before performing the two required Fourier integrals we remark that

$$\int\frac{d^dk}{(2\pi)^d}G_\Psi(\mathbf{q},\mathbf{k};\omega)=\frac{C_d\omega^{1-d/2}}{\sigma_\Psi+C_d\omega^{1-d/2}}\frac{1}{\mathbf{q}^2+\omega}, \tag{36}$$

so that, returning to the time domain,

$$\int\frac{d^dk}{(2\pi)^d}G_\Psi(\mathbf{q},\mathbf{k};s)=\int\frac{d\omega}{2\pi i}e^{\omega s}G_\Psi(\mathbf{q},\mathbf{k};\omega) \tag{37}$$

by folding the integration path around the cut on the negative half-axis, we have

$$\int\frac{d^dk}{(2\pi)^d}G_\Psi(\mathbf{q},\mathbf{k};s)=\frac{\sin[\pi(1-d/2)]}{\pi}\frac{(4\pi)^{d/2}}{\Gamma(1-d/2)}\int_0^{+\infty}dx e^{-xs}\frac{1}{x^{1-d/2}+2\cos[\pi(1-d/2)]+x^{d/2-1}}. \tag{38}$$

The last step is to compute the integral

$$K_u=\int_0^\infty\int_0^\infty dxdy\frac{(xy)^u}{[1+2\cos(2u\pi)x+x^2][1+2\cos(2u\pi)y+y^2](x^{1/u}+y^{1/u})}, \tag{39}$$

with $u=1-d/2$. We find that

$$K_{1-d/2}=\frac{K^{(1)}}{\varepsilon}+K^{(2)}+O(\varepsilon),\quad \varepsilon\to 0, \tag{40}$$

where we have dropped the leading UV divergence. A simple asymptotic expansion gives $K^{(1)}=4\pi/\sqrt{3}$. By rewriting σ and g in terms of renormalized quantities and developing first in powers of g_R then in ε the equation of state can finally be written as

$$\sigma_R\Psi+A^{-1/2}g_R\Psi^2-2g_R^2\Psi(\sigma_R+2A^{-1/2}g_R\Psi)\left\{B+\frac{K^{(2)}}{K^{(1)}}-3\ln(\sigma_R+2A^{-1/2}g_R\Psi)\right\}=0, \tag{41}$$

where $A=\sqrt{3}\Gamma(1/3)^3/(8\pi^3)$, and B is a numerical constant. The diverging terms of order $1/\varepsilon$ disappear from Eq. (32) to Eq. (41) as expected. As we look for the first correction to the mean-field result $\Psi=|\sigma_R|/g$, we can write $\Psi=(|\sigma_R|/g)(1+cg_R^2)$ and develop Eq. (41) to g_R^2 order to determine the coefficient c. Assuming that $|\sigma_R|\to 0$ at criticality, one finally finds

$$\Psi = C_\varepsilon|\sigma_R|^{1-3\varepsilon/8}, \quad C_\varepsilon \simeq \frac{\sqrt{\Gamma(1/3)}}{2\pi\sqrt{\varepsilon}}. \tag{42}$$

This nicely confirms the one-loop expansion of the order parameter exponent $\beta = 1-3\varepsilon/8$ and further yield the amplitude to leading ε order.

E. Finite-size scaling analysis

It is instructive to investigate some finite-size properties of the transition and to compare with what one would obtain for a usual DP process. In particular we would like to know, in a finite-size system, what the lifetime of the active state is. We follow the route outlined by Janssen *et al.* [12]. We split the order parameter field $\psi(\mathbf{q},t)$ into

$$\psi(\mathbf{q},t) = \Psi(t) + \phi(\mathbf{q},t), \tag{43}$$

where $\Psi(t) = \psi(\mathbf{q}=\mathbf{0},t)$, which defines $\phi(\mathbf{q},t)$. With periodic boundary conditions, the Fourier modes are $\mathbf{q} = (2\pi/L)(n_1,\dots,n_d)$, $n_\mu \in \{0,\dots,L-1\}$. The next step is to rewrite the action $S[\bar{\psi},\psi]$ in terms of the newly introduced fields. Our interest goes to the homogeneous mode in the steady-state, so that by integrating out the $\mathbf{q}\neq\mathbf{0}$ modes we obtain an effective action S_{eff} for the $\mathbf{q}=\mathbf{0}$ mode $\Psi(t)$ only. Of course, the latter procedure can only be performed within the framework of a (renormalized) perturbation expansion. The one-loop result reads

$$S_{\text{eff}}[\bar{\Psi},\Psi] = \int dt[L^d\bar{\psi}\dot{\Psi} + \sigma_R\bar{\psi}\Psi + g_R\bar{\Psi}\Psi(\Psi-\bar{\Psi})]. \tag{44}$$

We prefer working in rescaled fields and variables,

$$t = \frac{L^{3d/2}}{g_R}\tau, \quad \Psi(t) = L^{-d/2}\Phi(\tau). \tag{45}$$

The action S_{eff} for Φ is equivalent to the probability density function $P(\Phi,\tau)$ of Φ satisfying the following Fokker-Planck equation:

$$\partial_\tau P(\Phi,\tau) = \frac{\partial}{\partial\Phi}[(\gamma\Phi+\Phi^2)P] + \frac{\partial^2}{\partial\Phi^2}(\Phi P), \tag{46}$$

with $\gamma \equiv (\sigma_R/g_R)L^{d/2}$. It is well known [12] that the smallest nonzero eigenvalue of Eq. (46) is

$$\omega \sim |\gamma|^3 e^{-\gamma^2/2}, \tag{47}$$

which decays as e^{-L^d} as $L\to\infty$. Hence the lifetime of the active state goes to infinity in the thermodynamic limit. This parallels exactly what occurs in a usual directed percolation process. This adds an important feature that our model shares with a common absorbing-state transition.

IV. ANALYSIS OF THE FERMIONIC CASE

A. Heuristic approach

In contrast to the bosonic case, there is no firmly established easy-to-use field-theoretic formalism to cope with the fermionic constraint. It was recently shown how to incorporate the exclusion constraint in a field-theory of a bosonic type [13]. But the resulting action has exponential interaction terms that are difficult to exploit analytically, except by expanding them. This would lead us back to analysis of the bosonic theory, dropping terms potentially relevant in dimension 1. It is interesting to note that the action that incorporates the hardcore constraint does not have the time-reversal symmetry that the bosonic action possesses.

An alternative approach suggested by Cardy [14] consists in working directly in dimension 1 and in observing that the world lines of hardcore bosons behave as those of free fermions, at least between branching and annihilation events. This is equivalent to studying the effect of branching at the annihilation fixed point that corresponds (in the present inhomogenous case as well) to an infinite rate annihilation of particles meeting at the origin. We briefly recall the steps leading to a fermionic field theory accounting for the hardcore exclusion.

B. From the master equation to a quantum spin chain

Let $\mathsf{n}\equiv\{n_i\}$ denote a general configuration of the local occupation numbers n_i ($n_i=0$ or 1). The time evolution of the probability $P(\mathsf{n};t)$ to find the system in the microstate n at time t evolves according to a master equation

$$\frac{dP(\mathsf{n};t)}{dt} = \sum_{\mathsf{n}'}[W(\mathsf{n}'\to\mathsf{n})P(\mathsf{n}';t) - W(\mathsf{n}\to\mathsf{n}')P(\mathsf{n};t)], \tag{48}$$

where $W(\mathsf{n}'\to\mathsf{n})dt$ is the probability that the system goes from state n' to state n between t and $t+dt$. Setting $|\Phi(t)\rangle = \Sigma_{\mathsf{n}}P(\mathsf{n};t)|\mathsf{n}\rangle$, the master equation is equivalent to an evolution equation

$$\frac{d|\Phi\rangle}{dt} = -\hat{H}|\Phi\rangle, \tag{49}$$

where the stochastic evolution operator $\hat{H}$ encodes the microscopic dynamics defining the model. The reader is referred to Schütz [15] for a review of the properties of such operators. When a site has occupation number n, one defines the spin variable $s=2n-1$, which is the eigenvalue of the operator $\sigma^z = 2\hat{n}-1$. In the spin language, the evolution operator usually takes the form of conventional quantum spin chain Hamiltonian in an external (possibly complex) field. For reactions involving three particles or more, three-spin terms are present.

C. From the spin chain to a fermionic theory

Spin operators are known to be convenient to describe hardcore bosons. The Jordan-Wigner transformation allows one to express physical observables in terms of fermionic operators. It can be shown that a physical observable is obtained by determining the vacuum expectation value of products of fermionic operators weighted by a factor $e^{-\tilde{H}t}$, where

$$\tilde{H}=e^{\sum_i \sigma_i^-}\hat{H}e^{-\sum_i \sigma_i^-}. \qquad (50)$$

Hence $\tilde{H}$ is obtained from $\hat{H}$ by substituting the σ^+ by $\sigma^+ - \sigma^- + 1 - 2\hat{n}$. At this stage, the spin operators are replaced by their fermionic expressions. In order to build up a field theory, one simply replaces (after normal ordering) the fermionic operators by their Grassmann eigenvalues (this actually works for interaction terms made of an even number of fermionic operators, and odd powers must be dealt with greater care [16]). The average density is the average value of the Grassmann field associated to the annihilation operator weighted by $\exp(-S)$, where S is the fermionic action that depends on a pair of Grassmann fields.

D. Action and scaling analysis

In the fermionic case, only one particle is allowed to sit on site $x=0$; if an offspring is produced, it is placed on the neighboring site. Similarly, coagulation occurs between a particle located at $x=0$ and one of its two nearest neighbors. Power counting performed on the action

$$S[\overline{\psi},\psi]=\int dx\,dt\{\overline{\psi}[\partial_t+D\sigma\delta(x)-D\partial_x^2]\psi + Dg\,\delta(x)\overline{\psi}\psi(\partial_x\psi-\partial_x\overline{\psi})\} \qquad (51)$$

now indicates that $g\sim a^{1/2}$ and hence is irrelevant. Mean-field applies, hence in $d=1$, $\beta=1$. Let us stress that we have deliberately omitted nonlocal factors in the fermionic action. Following the analysis presented in Ref. [14], we know that they do not enter the scaling analysis though they will introduce nontrivial symmetry factors.

V. DISCUSSION AND PROSPECTS

We have shown that the hardcore constraint may be of crucial importance on the universality class of a transition to an absorbing state occurring in a nonequilibrium steady state. This is all the more surprising as the density approaches zero as one gets closer to the critical point, so that the probability of having several particles piled up on a site (in the bosonic version) must be extremely low. Although it is a well-understood fact that biased diffusion of hardcore particles exhibits a scaling behavior (described by the noisy Burgers equation) quite different from the purely diffusive bosonic counterpart, in the absence of any drift, it is hard to build up an intuitive picture that would fit into our findings. Similarly, it is well known that at a finite density, the typical displacement of a *tagged* hardcore particle increases with time as $t^{1/4}$, as opposed to the Brownian motion $t^{1/2}$ satisfied by bosonic particles. Yet, as far as collective behavior is concerned we know of very few properties specific to either family of particles (remember particles are indistinguishable). Admittedly, not being able to coin a heuristic picture does not mean there exists none, but it could also be an indication that the explanation lies in some "mathematical property" yet to identify, as was the case in BARWE. One could show that the time-reversal symmetry (12) satisfied by the actions describing bosonic particles of DP and of our model does not extend to fermionic particles. Could it play the role of such a "mathematical property?"

We would like to conclude on the difficulty of performing numerical simulations of our process. Besides the usual difficulties associated to absorbing states and to phase transitions, our system possesses a relaxation time that diverges much more sharply with $\lambda-\lambda_c$ than in an ordinary absorbing-state transition [in mean field it diverges as $(\lambda-\lambda_c)^{-2}$ as opposed to $(\lambda-\lambda_c)^{-1}$ for an ordinary DP process]. However, a numerical simulation would be necessary to investigate the boson vs fermion issue. Indeed a finite-size scaling analysis gives access to the ratio β/ν [using the scaling ansatz $\Psi(L,\sigma)=L^{-\beta/\nu}\mathcal{F}(L^{1/\nu}\sigma)$]. This is precisely the quantity that we predict exactly in the bosonic case ($\beta/\nu=1/2$) and that could be compared with mean field or fermionic value ($\beta/\nu=1$). A numerical analysis investigating those issues should be our next step.

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APPENDIX

In this appendix we would like to show how the original reaction-diffusion process can be described in terms of a non-Markovian single-site process taking place at the origin of the lattice. We shall integrate out the degrees of freedom describing pure diffusion on the sites $i\neq 0$ and find the resulting effective interaction for the number of particles at the origin at time t. Our motivations for doing this are twofold. First, since interactions are taking place at the origin only, it should prove more illuminating to have a process defined there in which only the free diffusion away from the origin has been incorporated as an effective interaction. Second, the renormalization group and scaling analyses rely on the existence of a continuous limit. However, due to the very singular role played by the origin, it could be that special renormalization has to be done for the field defined at the origin. In that respect, working directly with an interacting theory defined on the origin only will answer those worries.

For notational simplicity, we restrict the presentation to the one-dimensional case: the particles diffuse on a lattice of unit spacing with sites at $-L,-L+1,\ldots,L$. We adopt reflecting boundary conditions and set D as the diffusion constant. We consider an assembly of independently diffusing particles. The action describing their dynamics is

$$S=\int dt\Bigg[\sum_{i=1}^{L}\overline{\phi}_i\partial_t\phi_i-D\sum_{i=2}^{L-1}\overline{\phi}_i(\phi_{i-1}-2\phi_i+\phi_{i+1}) - D\overline{\phi}_1(\phi_0-2\phi_1+\phi_2)-D\overline{\phi}_L(\phi_{L-1}-\phi_L)\sum_{i=-1}^{-L}\overline{\phi}_i\partial_t\phi_i$$

$$-D\sum_{i=-2}^{-L+1}\overline{\phi}_i(\phi_{i-1}-2\phi_i+\phi_{i+1})-D\overline{\phi}_1(\phi_0-2\phi_{-1}+\phi_{-2})-D\overline{\phi}_{-L}(\phi_{-L+1}-\phi_{-L})\overline{\phi}_0\partial_t\phi_0-D\overline{\phi}_0(\phi_{-1}-2\phi_0+\phi_1). \quad \text{(A1)}$$

We want to trace out all the fields $\{\overline{\phi}_i(t),\phi_i(t)\}_{i\neq 0}$ describing diffusion away from the origin. We split the action in the form

$$S=S_0+S_1^++S_1^-+S_2^++S_2^-, \quad \text{(A2)}$$

with $S_0=\int dt\overline{\phi}_0\partial_t\phi_0+2d\overline{\phi}_0\phi_0$, $S_1^{\pm}=-\int dtD(\overline{\phi}_{\pm 1}\phi_0+\overline{\phi}_0\phi_{\pm 1})$, and

$$S_2^+=\int dt\left[\sum_{i=1}^{L}\overline{\phi}_i\partial_t\phi_i-D\sum_{i=2}^{L-1}\overline{\phi}_i(\phi_{i-1}-2\phi_i+\phi_{i+1})-D\overline{\phi}_1(-2\phi_1+\phi_2)-D\overline{\phi}_L(\phi_{L-1}-\phi_L)\right]$$
$$=\int dt\left[\sum_{i=1}^{L}\overline{\phi}_i\partial_t\phi_i-D\sum_{i,j=1}^{L}\overline{\phi}_iA_{ij}\phi_j\right], \quad \text{(A3)}$$

with A_{ij} an $L\times L$ symmetric matrix,

$$A=\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & & 1 & -1 \end{pmatrix}. \quad \text{(A4)}$$

The analog for $i=-L,\dots,-1$ defines S_2^-. There exists a unitary matrix U such that

$$\sum_{i,j=1}^{L}U_{qi}^{-1}A_{ij}U_{jq'}=\delta_{qq'}\lambda_q. \quad \text{(A5)}$$

We define $\psi_q(t)=\Sigma_jU_{jq}\phi_j$ and $\overline{\psi}_q(t)=\Sigma_jU_{jq}\overline{\phi}_j$. Then $S_2^+=\int dt\Sigma_q\overline{\psi}_q(\partial_t-D\lambda_q)\psi_{-q}$. Introducing the Fourier transforms

$$\psi_q(\omega)=\int dte^{i\omega t}\psi_q(t),\quad \overline{\psi}_q(\omega)=\int dte^{i\omega t}\overline{\psi}_q(t), \quad \text{(A6)}$$

we find

$$S_2^+=\int\frac{d\omega}{2\pi}\sum_q\overline{\psi}_{-q}(-\omega)(-i\omega-D\lambda_q)\psi_q(\omega),$$
$$\lambda_q=-2(1-\cos q). \quad \text{(A7)}$$

We also express S_1^+ in terms of $\overline{\psi},\psi$ and we perform explicitly the integration over those fields, with $\overline{\phi}_0,\phi_0$ as parameters. We obtain the total effective action

$$S_{\text{eff}}[\overline{\phi}_0,\phi_0]=\int dt\left[\overline{\phi}_0\partial_t\phi_0+2D\overline{\phi}_0\phi_0+2D^2\int_{-\infty}^{t}dt'\sum_q U_{1q}^2e^{D\lambda_q(t-t')}\overline{\phi}_0(t)\phi_0(t')\right]. \quad \text{(A8)}$$

Solving the diagonalization problem yields $U_{1q}=\sqrt{(2/L)}\sin q$ with q solution of $\tan Lq=\cot q/2$, $0<q<\pi$ (there are exactly L solutions). We now turn to the interpretation of the memory kernel appearing in the effective action. The Green function associated with the matrix A is

$$G_{ij}(t)=\sum_q U_{jq}U_{qi}^{-1}e^{D\lambda_q t}, \quad \text{(A9)}$$

which is the probability to be at site j at time t given the walker started at site i at time 0. The quantity $DG_{11}(t)$ is the flow of probability from 1 to 0 at time t, for a walker that started from site 1 at time 0. Since this flow is irreversible, $DG_{11}(t)$ is also the probability that the first visit to 0 takes place exactly at time t given the walker started from site 1 at time 0. Standard random walk theory allows to find the Laplace transform of $DG_{11}(t)$,

$$D\hat{G}_{11}(p)=\frac{p}{2D}-\frac{1}{2D}\frac{1}{G_0(p)}, \quad \text{(A10)}$$

with $G_0(p)=\int_{-\pi}^{\pi}(dq/2\pi)[1/p+2D(1-\cos q)]$ the Laplace transform of the return to the origin probability of a simple random walk. Returning to the time domain, one finds that

$$S_{\text{eff}}[\overline{\phi}_0,\phi_0]=\int dt\int_{-\infty}^{t}dt'\Gamma(t-t')\overline{\phi}_0(t)\phi_0(t'), \quad \text{(A11)}$$

with the Laplace transform of the memory kernel

$$\hat{\Gamma}(p)=\frac{1}{G_0(p)}. \quad \text{(A12)}$$

We are thus left with the study of the zero-dimensional time dependent process $\phi(t)$ whose dynamics is encoded in the following action:

$$S[\overline{\phi}_0,\phi_0]=\int dtdt'\overline{\phi}(t)\Theta(t-t')\Gamma(t,t')\phi(t')+\text{reaction terms}. \quad \text{(A13)}$$

It is well known that in the continuous limit (now generalizing to $d<2$ space dimensions)

$$G_0(p)=\frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}}p^{d/2-1},\quad p\to 0, \quad \text{(A14)}$$

and we shall henceforth adopt this limiting form, supplemented by a high-frequency cutoff Ω. This is equivalent to using the memory kernel

$$\Gamma(t,t')=\Theta(t-t')\frac{1}{[4\pi(t-t')]^{d/2}} \tag{A15}$$

with a short time cutoff Ω^{-1}. The quantity $G_0(p)/[1+\sigma G_0(p)]$ plays the role of the propagator in this zero-dimensional process. We now perform the power-counting procedures in units of the frequency cutoff Ω:

$$\phi(t)\sim\overline{\phi}(t)\sim\Omega^{d/4}, \quad g\sim\Omega^{1-3d/4}. \tag{A16}$$

We shall now perform a Wilson-type renormalization procedure. We trace out all degrees of freedom $\phi_>(\omega)$ with frequency in the interval $\Omega\geqslant|\omega|\geqslant\Omega/b$, where $b>1$ is a rescaling factor to be taken asymptotically close to 1. Of course the tracing out of those high-frequency degrees of freedom can only be performed perturbatively. The first nontrivial contributions give rise to effective couplings between the $\phi_<$ fields (defined for frequencies $|\omega|<\Omega/b$) exactly of the same form as the already present ones. Thus the theory defined at the coarse-grained scale Ω/b has new b-dependent couplings $\sigma(b)$ and $g(b)$. The recursion relation for the effective couplings defined at scale $b=e^l$ are

$$\frac{d\sigma}{dl}=\left(1-\frac{d}{2}-4g^2K_d\Omega^{-\varepsilon/2}\right)\sigma+\frac{2\Gamma\left(1-\dfrac{d}{2}\right)^2}{(4\pi)^{d+1}}g^2c_d\Omega^d, \quad \varepsilon=4-3d \tag{A17}$$

and

$$\frac{dg}{dl}=g\left(\frac{\varepsilon}{4}-8g^2K_d\Omega^{-\varepsilon/2}\right), \quad K_d=\frac{2\Gamma(1-d/2)^3}{(4\pi)^{(3d+2)/2}}. \tag{A18}$$

The latter equation possesses a single stable fixed point $g^{\star 2}K_d\Omega^{-\varepsilon/2}=\varepsilon/32$. The correlation time critical exponent thus has the expression

$$(z\nu)^{-1}=1-\frac{d}{2}-4g^{\star 2}K_d\Omega^{-\varepsilon/2}=1-\frac{d}{2}-\frac{\varepsilon}{8}=\frac{1}{3}+\frac{\varepsilon}{24}. \tag{A19}$$

One recovers in an elegant way the dimensional regularization results for the critical exponents.

No relevant effective interaction is generated: in particular, the one-loop graph renormalizing the propagator is proportional to

$$\int\frac{d\omega}{2\pi}\phi_<(-\omega)\phi_<(\omega) \times\int\frac{dw}{2\pi}\left|\frac{G_0(w)}{1+\sigma G_0(w)}\frac{G_0(\omega-w)}{1+\sigma G_0(\omega-w)}\right|. \tag{A20}$$

Expanding the terms in the integral in powers of ω produces only integer powers of ω, which correspond to irrelevant interactions. This establishes the absence of field renormalization in this problem.

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